JOURNAL OF FUNCTIONAL ANALYSIS 13, 173–184 (1973)

Global Solutions of the Cauchy Problem for the (Classical) Coupled Maxwell-Dirac Equations in one Space Dimension

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Communicated by Tosio Kato

Received October 18, 1972

The existence of global solutions of the Cauchy problem is proved for the Maxwell–Dirac equations coupled through the standard electromagnetic interaction. The proof depends on the conservation of charge and an a priori estimate on the electromagnetic potential. The technique also applies to the Dirac–Klein–Gordon equations with Yukawa coupling.

1. INTRODUCTION

The Cauchy problem for the coupled Maxwell–Dirac equations[1]

$$(-i\gamma^\mu \partial_\mu + m)\psi = g v^\mu \gamma_\mu \psi, \tag{1a}$$

$$\Box v_\mu = (\Delta - \partial_0{}^2) v_\mu = g \bar{\psi} \gamma_\mu \psi, \tag{1b}$$

$$\partial^\mu v_\mu = 0, \tag{1c}$$

has attracted some attention in recent years. Gross [1] has shown that in three space dimensions a unique solution exists locally. In a previous work [2] the author investigated the global problem showing that solutions of "cutoff" versions of Eqs. (1) [i.e., $g = g(\vec{x})$ and the Cauchy data having compact support] could be extended to all time. This led to the solution of the Cauchy problem for the true coupled Maxwell–Dirac equations in an arbitrary bounded region of four

[1] The v^μ's are the components of the electromagnetic (real) vector field, and ψ is the Dirac spinor field; i.e., ψ is a function from space–time into spin space. The positive definite inner product in spin space is denoted by $\psi^\dagger \psi$ and $\bar{\psi}$ denotes $\psi^\dagger \gamma^0$. The γ's are operators in spin space which satisfy $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ ($g^{00} = 1$, $g^{11} = -1$, $g^{\mu\nu} = 0$, $\mu \neq \nu$) and $\gamma^{0*} = \gamma^0$, $\gamma^{1*} = -\gamma^1$. All representations of operators satisfying these properties are unitarily equivalent. For this work, as usual, no specific choice is required.

dimensional space–time provided that the size of the coupling constant g or the Cauchy data is suitably restricted.

In this paper we shall prove that the Cauchy problem for Eqs. (1) in one space dimension has a unique global solution regardless of the size of g or the data. The local problem can be treated by either of the now standard techniques [3, 4] for handling abstract hyperbolic equations. The extension of this solution to all time is accomplished by a method which might be called "boot-strapping." Specifically, beginning with the conservation of charge, one shows successively that larger quantities remain finite. This leads ultimately to a bound on the norm of the local solution thus proving that it can be extended to all time.

In Sec. 2 the basic definitions will be given, and the local problem will be discussed. The global solution will be obtained in Sec. 3. A summary of the application of this technique to the coupled Dirac–Klein–Gordon equations (Yukawa interaction) will be given in Sec. 4.

2. The Local Problem

The solution spaces in which the Cauchy problem will be formulated will be described first. More details can be found in Refs. [1, 2].

Definition. Let D be the Hilbert space of spatially square integrable functions with values in spin space.[2] Denote the diagonal operator $(m^2 I - \Delta)^{1/2}$ by A. Then D_1 is defined to be $D(A) \subset D$ endowed with the norm

$$\| \psi \|_{D_1} = \| A\psi \|_2 .^3$$

Definition. Let M be the Hilbert space of square integrable functions $\binom{v}{\dot v}$ from E^1 with values in $\mathbb{R}^2 \oplus \mathbb{R}^2$. Denote the (2×2) diagonal operator $(-\Delta)^{1/2}$ by B. Then $M_2 = M_2{}^1 \oplus M_2{}^2$ is defined to be $D(B) \oplus L^2$ endowed with the norm

$$\left\| \begin{pmatrix} v \\ \dot v \end{pmatrix} \right\|_{M_{1/2}} = \{ \| Bv \|_2^2 + \| v \|_2^2 + \| \dot v \|_2^2 \}^{1/2}.$$

[2] In three space dimensions spin space is four dimensional while in one space dimension, as is the case in this work, it is two dimensional.

[3] The usual notations for function spaces [e.g., $L^p(E^1)$ with norm $\| \, \|_p$, $C_c{}^\infty(E^1)$ etc.] will be used for the appropriate direct sum analog of these spaces when there is no possibility of confusion.

Remark. Clearly D_1 and $M_{1/2}$ are complete and equivalent as Hilbert spaces to direct sums of the Sobolev space H^1. The present notation is essentially that of Refs. [1, 2]. The only difference is that we insist that the L^2 norm of the electromagnetic potential be finite in order to avoid technical complication in the definition of $M^1_{1/2}$ (otherwise it would consist of equivalence classes of functions modulo constants), as well as for computational reasons in what follows.

The Cauchy problem for Eqs. (1) can now be precisely stated in this Hilbert space setting. First, Eqs. (1) can be written in the more convenient vector-valued form,

$$\frac{d}{dt}\psi(t) = \left(\alpha\frac{\partial}{\partial x} + \beta m\right)\psi(t) + gV(t)\,\psi(t), \tag{2a}$$

$$\frac{d}{dt}\begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix} + g\begin{pmatrix} 0 \\ J(t) \end{pmatrix}, \tag{2b}$$

where

$$\alpha = -\gamma^0\gamma^1, \qquad \beta = -i\gamma^0, \qquad v(t) = \begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix}\; V(t) = i(v_0(t) - v_1(t)\alpha)$$

and

$$J(t) = \begin{pmatrix} J_0(t) \\ J_1(t) \end{pmatrix} = \begin{pmatrix} \psi^\dagger\psi(t) \\ \psi^\dagger\alpha\psi(t) \end{pmatrix}.$$

The Cauchy problem then is, for given data

$$\left(\psi^0, \begin{pmatrix} v^0 \\ \dot{v}^0 \end{pmatrix}\right) \in D_1 \oplus M_{1/2}$$

at some time t_0, to find functions ψ and v, with

$$t \to \left(\psi(t), \begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix}\right): (t_0, T) \to D_1 \oplus M_{1/2}$$

continuous, which satisfy, for $t_0 < t < T$, the integrated form of Eqs. (2):

$$\psi(t) = D(t - t_0)\psi^0 - g\int_{t_0}^{t} D(t - s)\, V(s)\,\psi(s)\, ds, \tag{3a}$$

$$\begin{pmatrix} v(t) \\ \dot{v}(t) \end{pmatrix} = M(t - t_0)\begin{pmatrix} v^0 \\ \dot{v}^0 \end{pmatrix} - g\int_{t_0}^{t} M(t - s)\begin{pmatrix} 0 \\ J(s) \end{pmatrix} ds, \tag{3b}$$

where

$$D(t) = e^{t(\alpha(\partial/\partial x) + \beta m))}$$

and

$$M(t) = \begin{pmatrix} \cos tB & B^{-1} \sin tB \\ -B \sin tB & \cos tB \end{pmatrix}$$

are the Dirac and Maxwell propagators. The integrals appearing in Eqs. (3) are to be interpreted in the strong Riemann sense. The solution is said to be global if T can be taken to be $+\infty$.

Gross' original proof [1] of local existence using Kato's results [3] can be greatly simplified in this case because the map

$$\left(\psi, \begin{pmatrix} v \\ \dot{v} \end{pmatrix}\right) \to \left(V\psi, \begin{pmatrix} 0 \\ J \end{pmatrix}\right): D_1 \oplus M_{1/2} \to D_1 \oplus M_{1/2}$$

is locally Lipschitzian.

THEOREM 2.1. *The (integrated form of the) coupled Maxwell–Dirac Eqs.* (3) *have a unique solution in* $D_1 \oplus M_{1/2}$ *for sufficiently small* $T - t_0$.

Proof. The fundamental result of Segal [4, Theorem 1, p. 343] can be applied directly. Now $D(t)$ and $M(t)$ are strongly continuous one-parameter unitary (resp. orthogonal) groups on D_1 (resp. $M_{1/2}$ with norm $\{\| Bv \|_2^2 + \| \dot{v} \|_2^2\}^{1/2}$). In addition, the first component of

$$M(t) \begin{pmatrix} v^0 \\ \dot{v}^0 \end{pmatrix}$$

is $\cos tBv^0 + B^{-1} \sin tB\dot{v}^0$ which as a map from $L^2 \oplus L^2 \to L^2$ is (e.g., by the spectral theorem) bounded by $k(1 + |t|)$ and strongly continuous. Thus $D(t) \oplus M(t)$: $D_1 \oplus M_{1/2} \to D_1 \oplus M_{1/2}$ is a continuous linear propagator as required in the above cited result of Segal. All that remains is to show that the nonlinear term is locally Lipschitzian.

To this end consider

$$\left\| \begin{pmatrix} V\psi \\ 0 \\ J \end{pmatrix} - \begin{pmatrix} \tilde{V}\tilde{\psi} \\ 0 \\ \tilde{J} \end{pmatrix} \right\|_{D_1 \oplus M_{1/2}} \leqslant \| V\psi - \tilde{V}\tilde{\psi} \|_{D_1} + \| J - \tilde{J} \|_{M_{1/2}^2}.$$

Now $\| V\psi - \tilde{V}\tilde{\psi} \|_{D_1} \leqslant C\{\| V\psi - \tilde{V}\tilde{\psi} \|_2 + \| D(V\psi - \tilde{V}\tilde{\psi})\|_2\}$ by standard results, where D is the strong L^2 spatial derivative. Since both v and ψ are in L^∞ (this follows in one dimension from the Sobolev inequality $\| f \|_\infty \leqslant C \| Df \|_2^{1/2} \| f \|_2^{1/2}$), the familiar rules of differential calculus can be used here for the strong derivatives. This along with straightforward algebra gives that the previous term is bounded by

$$\| v - \tilde{v} \|_\infty \| \psi \|_2 + \| \tilde{v} \|_\infty \| \psi - \tilde{\psi} \|_2 + \| D(v - \tilde{v})\|_2 \| \psi \|_\infty + \| D\tilde{v} \|_2 \| \psi - \tilde{\psi} \|_\infty$$

which proves the locally Lipschitz nature of the first component of the nonlinear term. The second component can be treated in a similar fashion.

$$\| J - \tilde{J} \|_{M_{1/2}^2} \leqslant \| \psi^\dagger\psi - \tilde{\psi}^\dagger\tilde{\psi} \|_2 + \| \psi^\dagger\alpha\psi - \tilde{\psi}^\dagger\alpha\tilde{\psi} \|_2$$

$$\leqslant (\| \psi \|_\infty + \| \tilde{\psi} \|_\infty) \| \psi - \tilde{\psi} \|_2 \leqslant C(\| \psi \|_{D_1} + \| \tilde{\psi} \|_{D_1}) \| \psi - \tilde{\psi} \|_2 ,$$

the last inequality following from the previously cited Sobolev inequality. These two estimates can then be combined to prove the locally Lipschitz nature of the nonlinear term and hence conclude the proof.

3. The Global Problem

The abstract result of Segal used in the last section states further that either $T = +\infty$ or the norm of the solution blows up for some finite time. Thus in order to show that the solution obtained in Theorem 2.1 can be extended to $+\infty$ all that needs to be done is to show that the $D_1 \oplus M_{1/2}$ norm of the solution is bounded for all finite time. As mentioned in the Introduction, this will be accomplished by beginning with the physically relevant law of conservation of charge. Although this is an ostensibly weak fact, it is positive definite and suffices to prove the result via the following steps. First one obtains an a priori estimate on $\| v(t)\|_\infty$. This along with the conservation law leads to a coupled set of inequalities for $\| \psi(t)\|_{D_1}$ and $\| Dv(t)\|_2$ which, with the aid of Gronwall's inequality, show that these quantities are finite. The result will then follow quickly. For convenience we take $t_0 = 0$ throughout.

Proposition 3.1. *Suppose that*

$$\left(\psi(t), \begin{matrix} v(t) \\ \dot{v}(t) \end{matrix}\right)$$

is a solution in $D_1 \oplus M_{1/2}$ of the coupled Maxwell–Dirac Eqs. (3). *Then the charge $\int_{E^1} \psi^\dagger\psi(x, t)\, dx = \| \psi(t)\|_2^2$ is conserved; i.e., $\| \psi(t)\|_2^2 = \| \psi(0)\|_2^2 = \| \psi^0 \|_2^2$.*

Proof. To begin, if the Cauchy data has sufficiently many derivatives in L^2, then the solution satisfies Eq. (1a) in the classical sense. A sketch of one way of seeing this is as follows. Take

$$\left(\psi^0, \begin{pmatrix} v^0 \\ \dot{v}^0 \end{pmatrix}\right) \in D_4 \oplus (H^4 \oplus H^3)$$

which is dense in $D_1 \oplus (M^1_{1/2} \oplus M^2_{1/2})$. A local existence theorem can be proved in this space along the same lines as Theorem 2.1. As in the proof of Theorem 2, p. 351 of Ref. [4], Eq. (2a) is satisfied in D_3. Furthermore $(d\psi/dt)(t)$ can be differentiated once more as in Theorem 3, p. 353 of the above work to obtain $(d^2\psi/dt^2)(t) \in D_2$. Thus for $t \in (0, T)$ $\psi(t)$, $(d\psi/dt)(t)$ and $(d^2\psi/dt^2)(t)$ are in $D_2 \cong H^2(E^1)$ and hence $\psi \in H^2(E^1 \times (0, T))$. By Sobolev's Imbedding Theorem ψ has a $C^1(E^1 \times (0, T))$ representative and it satisfies Eq. (1a) in the classical sense.

For such solutions the proof is well known in the physics literature and short enough to be repeated here. Multiplying Eq. (1a) on the left by $\bar{\psi} = \psi^\dagger\gamma_0$,

$$\bar{\psi}(-i\gamma^\mu\partial_\mu + m)\psi = gv^\mu\bar{\psi}\gamma_\mu\psi. \tag{4a}$$

On the other hand, taking the adjoint of Eq. (1a) one obtains

$$\psi^\dagger(i\gamma^0\overleftarrow{\partial}_0 - i\gamma^1\overleftarrow{\partial}_1 + m) = gv^\mu\psi^\dagger\gamma_\mu{}^*,$$

where the arrows indicate that the derivatives are applied to $\psi^\dagger$ which precedes them. Multiplying on the right by $\gamma^0\psi$ and permuting the γ matrices gives

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m)\psi = gv^\mu\bar{\psi}\gamma_\mu\psi. \tag{4b}$$

Subtracting (4a) from (4b) gives

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \frac{\partial}{\partial x}(\bar{\psi}\gamma^1\psi) = 0.$$

Integrating with respect to x one obtains

$$\frac{\partial}{\partial t}\int_{E^1} \psi^\dagger\psi(x, t)\, dx = \frac{\partial}{\partial t}\| \psi(t)\|_2^2 = 0, \tag{5}$$

because $\psi(x, t) \to 0$ as $| x | \to \infty$ (by the Riemann–Lebesgue lemma since $\psi(t) \in D_4 \subset D_1$ implies that $\hat{\psi}(t) \in L^1 \cap L^2$ for each t). This proves the result for smooth solutions and the result for $D_1 \oplus M_{1/2}$ solutions follows by continuity.

LEMMA 3.2. *For the solution given in Theorem* 2.1,

$$\| v(t)\|_\infty \leqslant a(1 + t), \tag{6}$$

where a is a constant depending upon the $D_1 \oplus M_{1/2}$ *norm of the Cauchy data* $(\psi^0, \binom{v^0}{\dot{v}^0})$ *given at time zero.*

Proof. For sufficiently smooth solutions

$$v(x, t) = \tfrac{1}{2}(v^0(x + t) + v^0(x - t)) + \tfrac{1}{2} \int_{x-t}^{x+t} \dot{v}^0(\xi)\, d\xi$$
$$+ \tfrac{1}{2} \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} J(y, \tau)\, dy\, d\tau.$$

Thus,

$$| v(x, t)| \leqslant \| v^0 \|_\infty + \tfrac{1}{2}t^{1/2} \| \dot{v}^0 \|_2 + \tfrac{1}{2} g \int_0^t \int_{-\infty}^{\infty} | J(y, \tau)|\, dy\, d\tau.$$

But $| J(y, \tau)| \leqslant$ const. $\psi^\dagger\psi(y, \tau)$, so that

$$| v(x, t)| \leqslant k(\| v^0 \|_{D_1} + \tfrac{1}{2}t^{1/2} \| \dot{v}^0 \|_2 + \tfrac{1}{2} gt \| \psi^0 \|_2^2)$$

thus proving the result for smooth solutions and by continuity for all solutions.

The main computational problem is included in the next result. A summarizing theorem stating the main result of this section will follow it.

LEMMA 3.3. *For the solution given in Theorem* 2.1, $\| Dv(t)\|_2$ *and* $\| \psi(t)\|_{D_1}$ *are locally bounded.*

Proof. From Eq. (3b)

$$\| Dv(t)\|_2 \leqslant \| Dv^0(t)\|_2 + g \int_0^t \left\| D \frac{\sin(t - s)B}{B} J(s) \right\|_2 ds$$
$$\leqslant \left\| \binom{v^0}{\dot{v}^0} \right\|_{M_{1/2}} + g \int_0^t \| J(s)\|_2\, ds$$

since D commutes with the Maxwell propagator and $D[\sin(t - s)B/B]$ is a bounded operator on L^2. Now straightforward algebraic calculations give that $\| J(s)\|_2 \leqslant \text{const} \| \psi(s)\|_4^2$. In one dimension we also have the Sobolev inequality $\| f \|_4 \leqslant \text{const} \| Df \|_2^{1/4} \| f \|_2^{3/4}$. Thus,

$$\| Dv(t)\|_2 \leqslant \left\| \begin{pmatrix} v^0 \\ \dot{v}^0 \end{pmatrix} \right\|_{M_{1/2}} + g \text{ const} \| \psi^0 \|_2^{3/2} \int_0^t \| D\psi(s)\|_2^{1/2} \, ds$$

$$\leqslant a + b \int_0^t \| \psi(s)\|_{D_1}^{1/2} \, ds, \tag{7}$$

where a, b are constants depending on the $D_1 \oplus M_{1/2}$ norm of the Cauchy data. Constants which are inessentially different will be confused in what follows.

Inequality (7) is the first of a pair coupling $\| Dv(t)\|_2$ and $\| \psi(t)\|_{D_1}$. For the second, beginning with Eq. (3a) we obtain

$$\| \psi(t)\|_{D_1} \leqslant \| \psi^0 \|_{D_1} + g \int_0^t \| D(t-s) \, V(s) \, \psi(s)\|_{D_1} \, ds$$

$$\leqslant \| \psi^0 \|_{D_1} + g \int_0^t \| V(s) \, \psi(s)\|_{D_1} \, ds$$

since the Dirac propagator is unitary on D_1. As in Theorem 2.1, the integrand can be estimated to yield

$$\| \psi(t)\|_{D_1} \leqslant \| \psi^0 \|_{D_1} + g \text{ const} \int_0^t \{\| v(s)\|_\infty \| \psi(t)\|_2 + \| Dv(s)\|_2 \| \psi(s)\|_\infty + \| v(s)\|_\infty \| D\psi(s)\|_2\} \, ds.$$

Applying the estimates obtained in Proposition 3.1 and Lemma 3.2 we obtain

$$\| \psi(t)\|_{D_1} \leqslant a + \int_0^t \{b(1+s) + d \| Dv(s)\|_2 \| \psi(s)\|_{D_1}^{1/2} + e(1+s) \| \psi(s)\|_{D_1}\} \, ds,$$

where in the middle term we have also used the Sobolev inequality $\| f \|_\infty \leqslant \text{const} \| Df \|_2^{1/2} \| f \|_2^{1/2}$. Thus,

$$\| \psi(t)\|_{D_1} \leqslant c(1+t)^2 + d \int_0^t \| Dv(s)\|_2 \| \psi(s)\|_{D_1}^{1/2} \, ds$$

$$+ e \int_0^t (1+s) \| \psi(s)\|_{D_1} \, ds. \tag{8}$$

The middle term of inequality (8) makes the direct application of Gronwall's lemma questionable. As it turns out the 1/2 power appearing in inequality (7) essentially reduces this to the first degree and hence makes it tractable. To see this take $f(t) = \| Dv(t)\|_2$ and $g(t) = \| \psi(t)\|_{D_1}^{1/2}$. Then inequalities (7) and (8) can be written in the notationally more convenient form

$$f(t) \leqslant a + b \int_0^t g(s)\, ds, \tag{9a}$$

$$g^2(t) \leqslant c(1+t)^2 + d \int_0^t f(s)\, g(s)\, ds + e \int_0^t (1+s)\, g^2(s)\, ds. \tag{9b}$$

In order to obtain coupled inequalities in $f^2(t)$ and $g^2(t)$, we square inequality (9a) to obtain.

$$\begin{aligned} f^2(t) &\leqslant a^2 + 2ab \int_0^t g(s)\, ds + b^2 \left(\int_0^t g(s)\, ds\right)^2 \\ &\leqslant a^2 + ab \int_0^t ds + ab \int_0^t g^2(s)\, ds + b^2 \int_0^t ds \cdot \int_0^t g^2(s)\, ds, \\ &\leqslant a^2 + abt + ab \int_0^t g^2(s)\, ds + b^2 t \int_0^t g^2(s)\, ds, \\ &\leqslant a(1+t) + b(1+t) \int_0^t g^2(s)\, ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} g^2(t) &\leqslant c(1+t)^2 + \frac{d}{2} \int_0^t f^2(s)\, ds + \frac{d}{2} \int_0^t g^2(s)\, ds + e \int_0^t (1+s)\, g^2(s)\, ds \\ &\leqslant c(1+t)^2 + d \int_0^t f^2(s)\, ds + e \int_0^t (1+s)\, g^2(s)\, ds. \end{aligned}$$

Adding and denoting $h(t) = f^2(t) + g^2(t)$ we obtain

$$h(t) \leqslant a(1+t)^2 + b(1+t) \int_0^t (1+s)\, h(s)\, ds.$$

Dividing both sides by $1 + t$ gives

$$\frac{h(t)}{1+t} \leqslant a(1+t) + b \int_0^t (1+s)^2 \frac{h(s)}{1+s}\, ds,$$

which by the Gronwall lemma implies that $h(t)/(1+t)$ is locally bounded proving the lemma.

THEOREM 3.4. *The (integrated form of the) coupled Maxwell–Dirac Eqs.* (3) *have a unique global solution in* $D_1 \oplus M_{1/2}$.

Proof. From the remarks at the beginning of the section all that needs to be shown is that the $D_1 \oplus M_{1/2}$ norm of the solution given in Theorem 2.1 remains finite for all t. In view of Lemma 3.3 all that remains to be proved is the finiteness of $\| v(t)\|_2$ and $\| \dot{v}(t)\|_2$.

$$\begin{aligned}
\left\|\begin{pmatrix} v(t) \\ \dot{v}(t)\end{pmatrix}\right\|_{L^2 \oplus L^2} &\leqslant \left\|\begin{pmatrix} v^0(t) \\ \dot{v}^0(t)\end{pmatrix}\right\|_{L^2 \oplus L^2} \\
&\quad + g \int_0^t \left\{ \left\| \frac{\sin(t-s)B}{B} J(s) \right\|_2 + \| \cos(t-s)\, BJ(s)\|_2 \right\} ds \\
&\leqslant \left\|\begin{pmatrix} v^0(t) \\ \dot{v}^0(t)\end{pmatrix}\right\|_{M_{1/2}} \\
&\quad + g \int_0^t \left\{ (t-s) \left\| \frac{\sin(t-s)B}{(t-s)B} J(s) \right\|_2 + \| \cos(t-s)\, BJ(s)\|_2 \right\} ds \\
&\leqslant c(1+t) \left\|\begin{pmatrix} v^0 \\ \dot{v}^0\end{pmatrix}\right\|_{M_{1/2}} + g \int_0^t (1+t-s) \| J(s)\|_2 \, ds \\
&\leqslant c(1+t) \left\|\begin{pmatrix} v^0 \\ \dot{v}^0\end{pmatrix}\right\|_{M_{1/2}} \\
&\quad + g \text{ const} \int_0^t (1+t-s) \| D\psi(s)\|_2^{1/2} \| \psi(s)\|_2^{3/2} \, ds.
\end{aligned}$$

The integrand is finite by Lemma 3.3 thus proving the Theorem.

In fact one further step in the "boot-strap" can be performed.

COROLLARY 3.5. *For the solution obtained in Theorem* 3.4, $\{\| B^2 v(t)\|_2^2 + \| B\dot{v}(t)\|_2^2\}^{1/2}$ *is finite if in addition the Cauchy data* $\binom{v^0}{\dot{v}^0} \in D(B^2) \oplus D(B)$.

Proof.

$$\begin{aligned}
&\{\| B^2 v(t)\|_2^2 + \| B\dot{v}(t)\|_2^2\}^{1/2} \\
&\quad \leqslant \{\| B^2 v^0(t)\|_2^2 + \| B\dot{v}^0(t)\|_2^2\}^{1/2} \\
&\qquad + g \int_0^t \{\| B^2 B^{-1} \sin(t-s)\, BJ(s)\|_2^2 + \| B \cos(t-s)\, BJ(s)\|_2^2\}^{1/2} \, ds \\
&\quad \leqslant \{\| B^2 v^0 \|_2^2 + \| B\dot{v}^0 \|_2^2\}^{1/2} + 2g \int_0^t \| BJ(s)\|_2 \, ds \\
&\quad \leqslant a + g \text{ const} \int_0^t \| D\psi(s)\|_2 \| \psi(s)\|_\infty \, ds \\
&\quad \leqslant a + b \int_0^t \| \psi(s)\|_{D_1}^2 \, ds.
\end{aligned}$$

4. Concluding Remarks

The preceding analysis applies equally well to the other important trilinear interaction encountered in field theory; namely the Dirac and Klein–Gordon equations coupled through a Yukawa interaction,

$$(-i\gamma^\mu \partial_\mu + m)\psi = g\varphi\psi, \tag{10a}$$

$$\Box\varphi + M^2\varphi = g\bar{\psi}\psi, \tag{10b}$$

where φ is a real scalar field. In this case the relevant solution space for the scalar field is the real, single component energy space $D_1 \oplus L^2$ associated with mass M. For more details see Ref. [5] and the references therein to the original work of Jörgens, Segal, and Strauss. The precise statement of the result which is possible is

Theorem 4.1. *The (integrated form of the) coupled Dirac–Klein–Gordon Eqs.* (10) *have a unique global solution in* $D_1 \oplus \{D_1 \oplus L^2\}$.

Sketch of the Proof. First the local result follows as in the proof of Theorem 2.1. The only change is a simplication in that the Klein–Gordon propagator is orthogonal on $D_1 \oplus L^2$. Furthermore the conservation of charge can be obtained by mimicking the proof of Proposition 3.1. The estimate on the scalar field follows as in Lemma 3.2 from the represention

$$\varphi(x, t) = \tfrac{1}{2}\{\varphi^0(x - t) + \varphi^0(x + t)\} + \tfrac{1}{2}\int_{x-t}^{x+t} J_0(M\sqrt{(x-\xi)^2 - t^2})\,\dot{\varphi}^0(\xi)\,d\xi$$

$$+ \tfrac{1}{2}\int_0^t \left[\int_{x-(t-\tau)}^{x+(t-\tau)} J_0(M\sqrt{(x-\xi)^2 - (t-\tau)^2})\,\bar{\psi}\psi(\xi, \tau)\,d\xi\right] d\tau$$

of sufficiently smooth solutions of Eq. (10b) and the uniform boundedness of the zeroth order Bessel function J_0. The technicalities of obtaining a coupled set of inequalities among $\|\psi(t)\|_{D_1}$ and $\|\varphi(t)\|_{D_1}$ which are tractable by the Gronwall lemma are essentially all contained in the proof of Lemma 3.3.

Finally, some comments should be made about the difficulties encountered in attempting to apply the preceding analysis to the physically most relevant case of three space dimensions. Although the conservation law obtains, an a priori bound on the nonspinor component (i.e., $\|v(t)\|_\infty$ or $\|\varphi(t)\|_\infty$) does not seem to be available by any of the standard methods. This quantity must therefore be included along with the energy norms in the coupled inequalities. However, this leads to an inequality of the form

$$f(t) \leqslant a + gb \int_0^t f^2(s)\, ds, \tag{11}$$

where f is the sum of the energy and sup norms and a and b are constants depending on the Cauchy data. Of course this leads to the boundedness of f only on an interval $(0, T)$, where T depends upon the size of g, a, and b. From another point of view this says that the solution exists throughout any bounded interval $(0, T)$ provided that the product of the coupling constant and Cauchy data is less than some number which depends upon T. But this is just the result proved by Gross [1] for the coupled Maxwell–Dirac equations. One notices then that it is precisely the availability of the a priori bound on $\| v(t)\|_\infty$ (or $\| \varphi(t)\|_\infty$) in one dimension which gives the global solution by reducing the power of the integrand in (11) to unity so that the standard Gronwall lemma applies.

Acknowledgment

The author is grateful to the referee for pointing out and helping to clarify a technical difficulty in the original manuscript.

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